$$\frac{\partial S}{\partial k_n} = 0, \ n = 1, 2, \ldots, N-1 \tag{30}$$

(There are not N equations in the Form (30) because there is one Equation (7) relating the k_n.) Thus, there are also 2N equations which can be solved for the 2N unknowns. The solution gives

$$p_n = p_{n-1} - \frac{(k_n{}^2 - 1)}{k_n{}^2} S, \quad n = 1, 2, \ldots, N-1 \tag{31}$$

$$k_1 = k_2 = \ldots = k_N \tag{32}$$

$$S = \frac{p}{N} \frac{K^{2/N}}{(K^{2/N} - 1)} \tag{33}$$

The residual pressures q_n and the required interferences for the shrink-fit assembly have yet to be found. The radial stress σ_{rn} at the radius r_n resulting from the bore pressure p is given by Equation (16a) with K replacing k_n, p replacing p_{n-1}, r_N replacing r_n, r_n replacing r, and $p_n = p_N = 0$. σ_{rn} becomes:

$$\sigma_{rn} = \frac{p}{K^2 - 1} (1 - k_{n+1}{}^2 \, k_{n+1}{}^2 \ldots k_N{}^2) \tag{34}$$

The pressure p_n is the sum of q_n and $(-\sigma_{rn})$. Therefore,

$$q_n = p_n - (-\sigma_{rn}) \tag{35}$$

The interference as manufactured, Δ at r , is given by

$$\frac{\Delta_n}{r_n} = \frac{-u_n(r_n)}{r_n} \quad \frac{u_{n+1}(r_n)}{r_n} \tag{36}$$

where

$u_n(r_n)$ = radial deformation at r_n of cylinder N due to the residual pressure q_n at r_n and the residual pressure q_{n-1} at r_{n-1}.

and

$u_{n+1}(r_n)$ = radial deformation at r_n of cylinder n+1 due to the residual pressure q_n at r_n and the residual pressure q_{n+1} at r_{n+1}.

Substituting the Expressions (35) for q_n into Expressions (17a) for the u_n and substituting the results into Equation (36), we find that Δ_n / r_n reduces to:

$$\frac{\Delta_n}{r_n} = \frac{2p}{NE} \tag{37}$$

The result p/2S given by Equation (33) is plotted in Figure 10 for various N. The limit curve is given by

$$\left(\frac{p}{2S}\right)_{limit} = \frac{K^2-1}{K^2} \tag{38}$$

at which limit the minimum shear stress becomes equal to -S at the bore in the inner cylinder.

Figure 10 has been obtained under the assumption that $\frac{\sigma_\theta - \sigma_r}{2}$ always gives the maximum shear stress. As pointed out by Berman[20], the maximum shear stress in a closed-end container* is given by $\frac{\sigma_z - \sigma_r}{2}$ when $\sigma_z > \sigma_\theta$. Therefore, it is important to know the limit to $\frac{p}{2S}$ for which σ_z becomes equal to σ_θ. σ_z is given by

$$\sigma_z = \frac{p}{K^2-1}$$

σ_θ is given by Equation (16b). Equating σ_θ at r_o to σ_z, we get the surprising result that the limit to $\frac{p}{2S}$ in this case is also given by Equation (38). Thus, the limit curve in Figure 4 has two meanings: it is the limit at which the minimum of the shear stress $\frac{\sigma_\theta - \sigma_r}{2}$ from residual pressures becomes equal to -S at the bore, and it is also the limit at which the bore shear stresses $\frac{\sigma_\theta - \sigma_r}{2}$ and $\frac{\sigma_z - \sigma_r}{2}$ become equal under the bore pressure p.

From the limit curve in Figure 10 and from Equation (38) it is found that

$$\lim_{K \to \infty} \left(\frac{p}{2S}\right) = 1 \tag{39}$$

Thus, the maximum pressure possible in a multi-ring container designed on the basis of static shear strength using ductile materials is p = 2S. For a ductile material with a tensile yield strength of 2S = 180,000 psi, this means that the maximum pressure is limited to 180,000 psi.

Fatigue Shear Strength Analysis

The optimum design of a multi-ring container having all rings of the same material and based on fatigue shear strength is found by an analysis similar to that conducted on the basis of static shear strength. Instead of minimizing S in Equation (30), σ given by the fatigue relation, Equation (12) is minimized, i.e.,

$$\frac{\partial \sigma}{\partial k_n} = 0, \; n = 1, 2, \ldots, N-1 \tag{40}$$

*Containers for hydrostatic extrusion generally are not closed-end containers. The effect of axial stress is included here for completeness.